Thermalized displaced and squeezed number states in the coordinate representation

1999 J. Phys. A: Math. Gen. 32 5037

(http://iopscience.iop.org/0305-4470/32/27/305)

J. Phys. A: Math. Gen. **32** (1999) 5037–5051. Printed in the UK PII: S0305-4470(99)03029-2

Thermalized displaced and squeezed number states in the coordinate representation

W-F Lu

CCAST (World Laboratory) PO Box 8730, Beijing 100080, People's Republic of China
and
Department of Applied Physics, Shanghai Jiao Tong University, Shanghai 200030, People's Republic of China†

E-mail: wenfalu@online.sh.cn

Received 26 March 1999

Abstract. Within the framework of thermofield dynamics, the wavefunctions of the thermalized displaced number and squeezed number states are given in the coordinate representation. Furthermore, the time evolution of these wavefunctions is considered by introducing a thermal coordinate representation, and we also calculate the corresponding probability densities, average values and variances of the position coordinate, some special cases of which are consistent with results in the literature.

1. Introduction

Displaced number states and squeezed number states are generalizations of coherent states and squeezed states of a harmonic oscillator, respectively [1]. The coherent state is constructed by displacing the ground state of the harmonic oscillator [2, 3], and the squeezed state by first squeezing the ground state and then further displacing it (sometimes, by first displacing and then squeezing, or by only squeezing) [4]. In these constructions, number states (also called Fock states in quantum field theory) of the harmonic oscillator taking the place of the ground state will correspondingly produce the displaced number state and squeezed number state. Thermalizing the displaced and squeezed number states, one can obtain the thermalized displaced number state and squeezed number state, which will be discussed in this paper. Evidently, thermalized coherent and squeezed states are the special cases of thermalized displaced number and squeezed number states. All the above states are interesting and important in physics.

As is well known, the coherent state can describe the coherent light, and its over-completeness gives rise to the coherent-state representation which is very useful in quantum optics, statistical physics, quantum field theory and particle physics, etc [5, 6]. This state mimics the motion of classical particles, and hence is also used for studying the Schrödinger cat states [7]. For the squeezed state, not only is it a minimum uncertainty state and similar to the classical motion, but also the quantum fluctuations in position can be suppressed at the expense of enhanced fluctuations in momentum [8], which is different from that of the coherent state. Therefore, the squeezed state has important technological applications in

† Mailing address.

0305-4470/99/275037+15\$30.00

quantum computation and sensitive measurement [9]. So far, the squeezed state has received a great deal of investigation [5, 10]. In the same way, a displaced number state follows the motion of a classical particle as well as keeping its shape in the course of the motion [1], and the squeezed number state can display a similar squeezed property to the squeezed state (equations (38)—(41) in [1]) and hence promises hopeful applications in optical spectroscopy, communications, molecular and solid state physics [11] (this reference dealt with the squeezed displaced number state†). Early in the 1950s, the displaced number state and the squeezed number state were proposed and studied with the help of the coordinate representation [12]. Since then, these states received a few further investigations [13, 14]. Recently, [1] reviewed these investigations, and gave the most general time-dependent wavefunctions and probability densities of them in the coordinate representation. Particularly, in view of the experimental realization of the optical and atomic squeezed (not displaced) states as well as the number states [9 (1996), 15], Nieto predicted that in the not too distant future, it would be possible to observe the displaced and squeezed number states [1].

On the other hand, thermal noises must exist somewhere, and thus the influence of the noises on the above-mentioned states has to be studied. Such an investigation is often realized by using density matrices and a master equation. However, within the framework of thermofield dynamics [16], a thermalizing operator acting on the states is also an important and useful way to introduce finite temperature effects [17, 18]. Both the density matrices and the thermofield-dynamics investigations give rise to a variety of thermal partners of the above-mentioned states, such as the thermalized coherent, squeezed, displaced number, and squeezed number states, the displaced thermalized state, squeezed thermalized state, and so on. Many properties of various thermal coherent and squeezed states have been studied by directly constructing a state vector [17–19] and other methods, such as characteristic function, density operators, Glauber's P-representation of a density operator, etc [14, 20–23] (most of [20, 21] were concerned with the coherent and squeezed thermalized states). The connections between these thermal states have been revealed in [18], and Fearn and Collett also gave the physical interpretations of these states [18]. Besides, for the thermal coherent state, Barnett and Knight discussed the independence of the Glauber's P-representation upon the order of displacing and thermalizing operators [17]. As for the thermalized displaced number and squeezed number states, there were few investigations of them, and just recently the thermalized squeezed number state (not displaced, different from the state in this paper) was considered with its characteristic function for analysing the influence of thermal noise on higher-order squeezing properties of it [24].

This paper will address the wavefunctions and position probability densities of the thermalized displaced number and squeezed number states (hereafter, two of these states will also imply that the thermalized coherent and squeezed states are their special cases). This problem has not, to our knowledge, been discussed in the literature, except for [20 (1993), 22 (1965)] in which the position probability density of coherent thermal state and squeezed thermal state (not including the number state) was given by Glauber's R-function and/or P-representation). However, this problem is certainly interesting and meaningful. The wavefunctions of the coherent, squeezed, displaced number and squeezed number states contain all the information about these states and hence describe these states completely. Therefore, the wavefunctions of the corresponding thermalized non-classical states will give the influence of finite temperature on the properties described by the zero-temperature wavefunctions, and can provide, at least, a quantum-mechanical intuitional understanding for us. Moreover, the coordinate representation of their density operators can be obtained from the finite-temperature wavefunctions and consequently these wavefunctions can equip a coordinate-representation

† The author thanks the referee for recommending this reference.

way for calculating the expectation values of all physical observables on the thermalized non-classical states, which is the most usual way in quantum mechanics. Additionally, the position density probability can give the probability density of magnetic component of electromagnetic fields [25, 22 (1965)].

Thermofield dynamics is a unique formalism for finding the wavefunctions for the thermal non-classical states. In this paper, within the framework of thermofield dynamics, we shall give wavefunctions of the thermalized displaced number and squeezed number states in terms of the position coordinate, consider their time evolution, and calculate the position probability densities. In order to do so, we shall first derive the wavefunction of the thermal vacuum in the coordinate representation, which was almost given in [8], and introduce a thermal coordinate representation in the next section. Then the wavefunctions of the thermalized displaced number and squeezed number states will be given in terms of the position coordinate in section 3. Section 4 will address the time evolution, the position probability densities, the position average values and variances of these states. We will give a conclusion at the end of this paper.

Note that, thermofield dynamics will be not introduced in this paper, however, good expositions of them can be found in [16]. Besides, although this paper will discuss a harmonic oscillator with a mass and constant frequency, by taking the mass as a unit one can get results which are usable for a one-mode electromagnetic field with the same frequency.

2. Thermal vacuum and thermal coordinate representation

In the fixed-time Schrödinger picture, for the quantum one-dimensional oscillator

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2 = \left(a^\dagger a + \frac{1}{2}\right)\hbar\omega \tag{1}$$

the ground state in the coordinate representation is the wavefunction

$$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left\{-\frac{m\omega}{2\hbar}x^2\right\} \tag{2}$$

where $p = -\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}x} \equiv -\mathrm{i}\hbar\partial_x$, m is the mass, ω the angular frequency, and

$$a = \frac{1}{\sqrt{2m\hbar\omega}}(\mathrm{i}p + m\omega x) \qquad a^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}(-\mathrm{i}p + m\omega x) \tag{3}$$

are the corresponding annihilation and creation operators, respectively. It is noticed that in [1], m, ω and $\hbar$ are all used as units. In order to consider thermal effects, thermofield dynamics introduces a copy of the physical oscillator equation (1) (called the tilde oscillator)

$$\tilde{H} = \frac{1}{2m}\tilde{p}^2 + \frac{1}{2}m\omega^2\tilde{x}^2 = \left(\tilde{a}^\dagger\tilde{a} + \frac{1}{2}\right)\hbar\omega \tag{4}$$

according to the tilde 'conjugation': $\widetilde{CO} \equiv C^*\tilde{O}$ [16]. Here, C is any coefficient appearing in expressions of quantities for the physical system, O any operator, the superscript $*$ means complex conjugation, and $\tilde{O}$ represents the corresponding operator for the tilde system. Exploiting the physical and tilde oscillators, one can have the thermal vacuum [16]

$$|0,\beta\rangle = T(\theta)|0,\tilde{0}\rangle \tag{5}$$

where, $|0,\tilde{0}\rangle = |0\rangle|\tilde{0}\rangle$ is the product of ground states of the physical and tilde oscillators, $\beta = \frac{1}{k_b T}$ with k_b the Boltzmann constant and T the temperature, and the unitary transformation $T(\theta)$ (called thermal transformation) is

$$T(\theta) = \exp\{-\theta(\beta)(a\tilde{a} - a^\dagger\tilde{a}^\dagger)\} \tag{6}$$

with

$$\tanh[\theta(\beta)] = \mathrm{e}^{-\beta\hbar\omega/2}.$$

Notice that any physical operator commutes with any tilde operator. Consequently, the thermal-vacuum average value agrees with canonical ensemble average in statistical mechanics.

It is evident that the thermal vacuum (5) is similar to the two-mode squeezed states discussed in [8] except for a minus difference between the exponents in equation (6) here and equation (37) there. Although the wavefunction of the two-mode squeezed state was given in the coordinate representation [8], here we still derive the position wavefunction of the thermal vacuum for the sake of both completeness and the establishment of the thermal coordinate representation. Substituting equation (3) into (6), one can read

$$T(\theta) = \exp\left\{\mathrm{i}\frac{\theta}{\hbar}(x\tilde{p} - \tilde{x}p)\right\} \tag{7}$$

with $\theta \equiv \theta(\beta)$. From appendix B.4 in [26], the last formula can be untangled as

$$T(\theta) = \exp\{-\tanh(\theta)\tilde{x}\partial_x\}\exp\{\ln[\cosh(\theta)](x\partial_x - \tilde{x}\partial_{\tilde{x}})\}\exp\{-\tanh(\theta)x\partial_{\tilde{x}}\}. \tag{8}$$

Using the following operator properties [27]

$$\mathrm{e}^{C\partial_y} f(y) = f(y + C) \tag{9}$$

and

$$\mathrm{e}^{Cy\partial_y} f(y) = f(y\mathrm{e}^C) \tag{10}$$

which are proved easily, we obtain the wavefunction of the thermal vacuum as

$$\begin{aligned}\langle\tilde{x}, x|0, \beta\rangle &= T(\theta)\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}}\exp\left\{-\frac{m\omega}{2\hbar}(x^2 + \tilde{x}^2)\right\} \\ &= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}}\exp\left\{-\frac{m\omega}{2\hbar}[(x\cosh(\theta) - \tilde{x}\sinh(\theta))^2 + (\tilde{x}\cosh(\theta) - x\sinh(\theta))^2]\right\}.\end{aligned} \tag{11}$$

When $\beta \to \infty$, $\langle\tilde{x}, x|0, \beta\rangle$ is reduced to $\langle\tilde{x}, x|0, \tilde{0}\rangle$. This expression (11) can be generalized to the Gaussian wavefunctional approach for equilibrium field theory in thermofield dynamics [28].

Such an expression of the thermal vacuum wavefunction equation (11) suggests the usefulness of introducing a thermal coordinate representation. In thermofield dynamics, for any operator of the physical or tilde oscillator Q, its thermal counterpart is defined as $Q_\beta \equiv T(\theta)QT^\dagger(\theta)$ [16]. In particular, for the fundamental canonical conjugate pairs $\{x, p = -\mathrm{i}\hbar\partial_x\}$ and $\{\tilde{x}, \tilde{p} = \mathrm{i}\hbar\partial_{\tilde{x}}\}$, the corresponding thermal operators are

$$\begin{aligned}x_\beta &\equiv T(\theta)xT^\dagger(\theta) = x\cosh(\theta) - \tilde{x}\sinh(\theta) \\ p_\beta &\equiv T(\theta)pT^\dagger(\theta) = p\cosh(\theta) - \tilde{p}\sinh(\theta)\end{aligned} \tag{12}$$

and

$$\begin{aligned}\tilde{x}_\beta &\equiv T(\theta)\tilde{x}T^\dagger(\theta) = \tilde{x}\cosh(\theta) - x\sinh(\theta) \\ \tilde{p}_\beta &\equiv T(\theta)pT^\dagger(\theta) = \tilde{p}\cosh(\theta) - p\sinh(\theta).\end{aligned} \tag{13}$$

Obviously, the thermal vacuum wavefunction equation (11) can be written as

$$\langle\tilde{x}, x|0, \beta\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}}\exp\left\{-\frac{m\omega}{2\hbar}(x_\beta^2 + \tilde{x}_\beta^2)\right\} \tag{14}$$

which is the same form with the wavefunction $\langle\tilde{x}, x|0, 0\rangle$. Noticing that the commutators $[x_\beta, p_\beta] = \mathrm{i}\hbar$, $[\tilde{x}_\beta, \tilde{p}_\beta] = -\mathrm{i}\hbar$ and $[O_\beta, \tilde{O}_\beta] = 0$ hold, one can set $p_\beta \equiv -\mathrm{i}\hbar\frac{\partial}{\partial x_\beta}$ and

$\tilde{p}_\beta \equiv i\hbar\frac{\partial}{\partial\tilde{x}_\beta}$, and establish a representation for the thermal oscillator, in which any object (operators, wavefunctions) can be expressed in terms of x_β, $\tilde{x}_\beta$, $\frac{\partial}{\partial x_\beta}$ and/or $\frac{\partial}{\partial\tilde{x}_\beta}$. In this paper, we shall call it thermal coordinate representation. Evidently, this representation is reached through the unitary thermal transformation $T(\theta)$ of the coordinate representation. When working in the representation, quantities will take similar forms to those in quantum mechanics, and hence it will simplify our derivation in this paper.

It is suitable here to mention a mathematical property and the physical sense of the thermal transformation equation (6). It is easily shown that the action of $T(\theta)$ on a function of the physical and tilde positions $\{x, \tilde{x}\}$ amounts to just the thermal coordinate $x_\beta, \tilde{x}_\beta$ taking the place of $x, \tilde{x}$, that is,

$$T(\theta)f(x,\tilde{x}) = f(x_\beta,\tilde{x}_\beta). \tag{15}$$

Thermal transformation is also called the thermalizing operator [18]. It describes the effect of a thermal reservoir in which a quantum harmonic oscillator immerses. From equation (5), we can say loosely that a thermalizing operator heats the ground state of a zero-temperature harmonic oscillator into a thermal vacuum with a finite temperature. In quantum optics, the thermalizing operator describes the action of a source which excites a one-mode electromagnetic field from its ground state to a chaotic state (thermalized radiation). Thus, in order to consider thermal noise, it is enough to perform the action of the thermalizing operator on the non-classical states mentioned in the last section. Next, we shall address them.

3. Thermalized displaced number and squeezed number state in the coordinate representation

Because both coherent and squeezed states are constructed with the displacing operator and/or squeezing operator acting on the ground state, there are three different states with squeezed effect: squeezed state (only the squeezing operator acting on the ground state), displaced squeezed state, and squeezed displaced state, which are all usually called squeezed state in the literature. In this paper, the terminology 'squeezed state' means only the displaced squeezed state, for which the action of the displacing operator follows that of the squeezing operator; so does the squeezed number state. However, when introducing a finite temperature effect, one still faces more choices about the orders among displacing, squeezing and thermalizing. A different order will lead to a different thermal non-classical state [18]. Nevertheless, if using thermal creation and annihilation operators to work, i.e., as is done in [17, 19], one can escape the order problems with the thermalizing operator. In this section, we shall introduce a finite temperature effect into the displaced number state and squeezed number state by using the thermal creation and annihilation operators with the vacuum $|0,\beta\rangle$ [17, 19] and then give their expressions in the coordinate representation. This construction is utterly to thermalize the displaced number and squeezed number states, namely, it gives the thermalized displaced number and squeezed number states, as one shall see later.

The thermal annihilation and creation operators with the thermal vacuum equation (5) are [16]

$$a_\beta = T(\theta)aT^\dagger(\theta) \qquad a_\beta^\dagger = T(\theta)a^\dagger T^\dagger(\theta) \tag{16}$$

and

$$\tilde{a}_\beta = T(\theta)\tilde{a}T^\dagger(\theta) \qquad \tilde{a}_\beta^\dagger = T(\theta)\tilde{a}^\dagger T^\dagger(\theta). \tag{17}$$

One can easily check that $a_\beta|0,\beta\rangle = 0$, $\tilde{a}_\beta|0,\beta\rangle = 0$ and $[a_\beta, a_\beta^\dagger] = [\tilde{a}_\beta, \tilde{a}_\beta^\dagger] = 1$. With the aid of thermal creation operators $a_\beta^\dagger$ and $\tilde{a}_\beta^\dagger$, one can construct normalized thermal number states

$$|n, m, \beta\rangle = \frac{1}{\sqrt{n!m!}} a_\beta^{\dagger n} \tilde{a}_\beta^{\dagger m} |0, \beta\rangle \tag{18}$$

with the closure relation

$$\sum_{n,m} |n, m, \beta\rangle\langle\beta, m, n| = 1. \tag{19}$$

The so-called displaced number state $|\alpha, n\rangle$ of the oscillator equation (1) is defined in Fock space as $|\alpha, n\rangle \equiv D(\alpha)|n\rangle$ and can have the following form in the coordinate representation [1]:

$$\begin{aligned}\langle x|\alpha, n\rangle = {} & \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} \exp\{-\mathrm{i}\alpha_1\alpha_2\} \\ & \times \exp\left\{-\frac{m\omega}{2\hbar}\left(x - \sqrt{\frac{2\hbar}{m\omega}}\alpha_1\right)^2 + \mathrm{i}\sqrt{\frac{2m\omega}{\hbar}}\alpha_2 x\right\} H_n\left[\sqrt{\frac{m\omega}{\hbar}}x - \sqrt{2}\alpha_1\right]\end{aligned} \tag{20}$$

with $\alpha = (\alpha_1 + \mathrm{i}\alpha_2)$ any complex number, $H_n[\ldots]$ the Hermite polynomials and the displacing operator

$$D(\alpha) = \mathrm{e}^{\alpha a^\dagger - \alpha^* a}. \tag{21}$$

In this definition, when the number state $|n\rangle$ is replaced by the ground state $|0\rangle$, the state $|\alpha, n\rangle$ is reduced to the usual coherent state $|\alpha\rangle$. Evidently, the state $|\alpha, n\rangle$ is constructed just with the displacing operator $D(\alpha)$ acting on the number state $|n\rangle$. Similarly, one can define the following state $|\alpha, n, \beta\rangle$:

$$|\alpha, n, \beta\rangle = D_\beta(\alpha)\tilde{D}_\beta(\alpha)|n, n, \beta\rangle \tag{22}$$

so as to introduce a finite temperature effect into the displaced number state. Here, the thermal displacing operators $D_\beta(\alpha)$ and $\tilde{D}_\beta(\alpha)$ are

$$D_\beta(\alpha) = \exp\{\alpha a_\beta^\dagger - \alpha^* a_\beta\} \tag{23}$$

$$\tilde{D}_\beta(\alpha) = \exp\{\tilde{\alpha}\tilde{a}_\beta^\dagger - \tilde{\alpha}^*\tilde{a}_\beta\} \tag{24}$$

respectively, which are generalizations of the displacing operator, $D(\alpha)$. Note that $\tilde{\alpha} = \alpha^*$ in this paper (of course, one can take $\tilde{\alpha}$ as another parameter independent of α). When $n = 0$, the state $|\alpha, n, \beta\rangle$ is just equation (11) with $\gamma = \alpha$ in Mann and Revzen [19] and equation (3.1) with $\varphi = \alpha$ in [17]. Employing the definitions (5), (16)–(18), we obtain

$$|\alpha, n, \beta\rangle = T(\theta)|\alpha, n\rangle|\tilde{\alpha}, n\rangle. \tag{25}$$

This equation indicates that the state $|\alpha, n, \beta\rangle$ is just the thermalized displaced number state. In the last equation, $|\tilde{\alpha}, n\rangle$ is the tilde displaced number state and can be obtained from equation (20) according to the tilde rules. When $n = 0$, the state $|\alpha, 0, \beta\rangle$ is the thermalized coherent state, being similar to equation (3.3) in [17]. Employing equations (15), (12) and (13), one can have

$$\begin{aligned}\langle\tilde{x}, x|\alpha, n, \beta\rangle = {} & \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \frac{1}{2^n n!} \exp\left\{-\frac{m\omega}{2\hbar}\left[\left(x_\beta - \sqrt{\frac{2\hbar}{m\omega}}\alpha_1\right)^2 + \left(\tilde{x}_\beta - \sqrt{\frac{2\hbar}{m\omega}}\alpha_1\right)^2\right]\right. \\ & \left. + \mathrm{i}\sqrt{\frac{2m\omega}{\hbar}}\alpha_2(x_\beta - \tilde{x}_\beta)\right\} H_n\left[\sqrt{\frac{m\omega}{\hbar}}x_\beta - \sqrt{2}\alpha_1\right] H_n\left[\sqrt{\frac{m\omega}{\hbar}}\tilde{x}_\beta - \sqrt{2}\alpha_1\right] \\ = {} & \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \frac{1}{2^n n!} \exp\left\{-\frac{m\omega}{2\hbar}\left[\left(x\cosh(\theta) - \tilde{x}\sinh(\theta) - \sqrt{\frac{2\hbar}{m\omega}}\alpha_1\right)^2\right.\right.\end{aligned} \tag{26}$$

$$+\left(\tilde{x}\cosh(\theta) - x\sinh(\theta) - \sqrt{\frac{2\hbar}{m\omega}}\alpha_1\right)^2\Bigg]$$
$$+\mathrm{i}\sqrt{\frac{2m\omega}{\hbar}}\alpha_2(\cosh(\theta)+\sinh(\theta))(x-\tilde{x})\Bigg\}$$
$$\times H_n\left[\sqrt{\frac{m\omega}{\hbar}}(x\cosh(\theta) - \tilde{x}\sinh(\theta)) - \sqrt{2}\alpha_1\right]$$
$$\times H_n\left[\sqrt{\frac{m\omega}{\hbar}}(\tilde{x}\cosh(\theta) - x\sinh(\theta)) - \sqrt{2}\alpha_1\right]. \tag{27}$$

The rhs of equation (26) is the wavefunction of the thermalized displaced number state in the thermal coordinate representation, and equation (27) is just the wavefunction in the coordinate representation. When $\beta \to \infty$, equation (27) is reduced to a product of the x- and $\tilde{x}$-function, each factor resembling equation (15) in [1].

Now we are in a position to discuss the thermalized squeezed number state. The squeezed number state $|\alpha, z, n\rangle$ of the oscillator equation (1) is constructed by using the squeezing operator $S(z)$ [4]

$$S(z) = \exp\{-\tfrac{1}{2}(z^*aa - za^\dagger a^\dagger)\} \tag{28}$$

which reads

$$|\alpha, z, n\rangle \equiv D(\alpha)S(z)|n\rangle. \tag{29}$$

Here, z is any complex constant. When $n = 0$, $|\alpha, z, 0\rangle$ is the usual squeezed state. From [1], the wavefunction of $|\alpha, z, n\rangle$ in the coordinate representation is (here in terms of our notations)

$$\langle x|\alpha, z, n\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \frac{(\sqrt{\mathcal{F}_3})^n}{\sqrt{\mathcal{F}_1 2^n n!}} \exp\{-\mathrm{i}\alpha_1\alpha_2\}$$
$$\times \exp\left\{-\frac{m\omega}{2\hbar}\mathcal{F}_2\left[x - \sqrt{\frac{2\hbar}{m\omega}}\alpha_1\right]^2 + \mathrm{i}\sqrt{\frac{2m\omega}{\hbar}}\alpha_2 x\right\}$$
$$\times H_n\left[\sqrt{\frac{m\omega}{\hbar}}(\mathcal{F}_4)^{-1}\left(x - \sqrt{\frac{2\hbar}{m\omega}}\alpha_1\right)\right] \tag{30}$$

with $z = z_1 + \mathrm{i}z_2 = r\mathrm{e}^{\mathrm{i}\phi}$, $\mathcal{S} = \cosh(r) + z_1\sinh(r)/r$ and $\kappa = z_2\sinh(r)/(2r\mathcal{S})$. In equation (30), for the convenience of comparison later, we adopted the notations $\mathcal{F}$ in [1], that is,

$$\mathcal{F}_1 = \mathcal{S}(1+\mathrm{i}2\kappa) \qquad \mathcal{F}_2 = \frac{1}{\mathcal{S}^2(1+\mathrm{i}2\kappa)} - \mathrm{i}2\kappa$$
$$\mathcal{F}_3 = \frac{1-\mathrm{i}2\kappa}{1+\mathrm{i}2\kappa} \qquad \mathcal{F}_4 = \mathcal{S}(1+4\kappa^2)^{\frac{1}{2}}. \tag{31}$$

In analogy with the definition of the squeezed number state, we introduce the thermal squeezing operator

$$S_\beta(z) = \exp\{-\tfrac{1}{2}(z^*a_\beta a_\beta - za_\beta^\dagger a_\beta^\dagger)\} \qquad \tilde{S}_\beta(z) = \exp\{-\tfrac{1}{2}(\tilde{z}^*\tilde{a}_\beta\tilde{a}_\beta - \tilde{z}\tilde{a}_\beta^\dagger\tilde{a}_\beta^\dagger)\} \tag{32}$$

and define the following state $|\alpha, z, n, \beta\rangle$

$$|\alpha, z, n, \beta\rangle \equiv D_\beta(\alpha)\tilde{D}_\beta(\alpha)S_\beta(z)\tilde{S}_\beta(z)|n, n, \beta\rangle \tag{33}$$

to introduce a finite temperature effect. Note that $\tilde{z} = z^*$ in this paper (of course, one can take $\tilde{z}$ as another parameter independent of z). It is easily shown that

$$|\alpha, z, n, \beta\rangle = T(\theta)|\alpha, z, n\rangle|\tilde{\alpha}, \tilde{z}, n\rangle \tag{34}$$

with $|\tilde{\alpha}, \tilde{z}, n\rangle$ the tilde version of $|\alpha, z, n\rangle$. Evidently, the last equation indicates that the state $|\alpha, z, n, \beta\rangle$ is just the thermalized displaced number state. When $n = 0$, $|\alpha, z, 0, \beta\rangle$ is just the thermalized squeezed state, being similar to equation (11) in Kireev *et al* [19]. Employing equations (15), (30), (12) and (13), we obtain

$$\begin{aligned}
\langle \tilde{x}, x|\alpha, z, n, \beta\rangle = {}& \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \frac{1}{|\mathcal{F}_1|2^n n!} \exp\left\{ -\frac{m\omega}{2\hbar}\left[\mathcal{F}_2\left(x_\beta - \sqrt{\frac{2\hbar}{m\omega}}\alpha_1\right)^2\right.\right. \\
& \left.\left. +\mathcal{F}_2^*\left(\tilde{x}_\beta - \sqrt{\frac{2\hbar}{m\omega}}\alpha_1\right)^2\right] + \mathrm{i}\sqrt{\frac{2m\omega}{\hbar}}\alpha_2(x_\beta - \tilde{x}_\beta)\right\} \\
& \times H_n\left[\sqrt{\frac{m\omega}{\hbar}}(\mathcal{F}_4)^{-1}\left(x_\beta - \sqrt{\frac{2\hbar}{m\omega}}\alpha_1\right)\right] H_n\left[\sqrt{\frac{m\omega}{\hbar}}(\mathcal{F}_4)^{-1}\left(\tilde{x}_\beta - \sqrt{\frac{2\hbar}{m\omega}}\alpha_1\right)\right]
\end{aligned} \tag{35}$$

$$\begin{aligned}
= {}& \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \frac{1}{|\mathcal{F}_1|2^n n!} \exp\left\{ -\frac{m\omega}{2\hbar}\left[\mathcal{F}_2\left(x\cosh(\theta) - \tilde{x}\sinh(\theta) - \sqrt{\frac{2\hbar}{m\omega}}\alpha_1\right)^2\right.\right. \\
& \left. +\mathcal{F}_2^*\left(\tilde{x}\cosh(\theta) - x\sinh(\theta) - \sqrt{\frac{2\hbar}{m\omega}}\alpha_1\right)^2\right] \\
& \left. +\mathrm{i}\sqrt{\frac{2m\omega}{\hbar}}\alpha_2(\cosh(\theta) + \sinh(\theta))(x - \tilde{x})\right\} \\
& \times H_n\left[\sqrt{\frac{m\omega}{\hbar}}(\mathcal{F}_4)^{-1}\left(x\cosh(\theta) - \tilde{x}\sinh(\theta) - \sqrt{\frac{2\hbar}{m\omega}}\alpha_1\right)\right] \\
& \times H_n\left[\sqrt{\frac{m\omega}{\hbar}}(\mathcal{F}_4)^{-1}\left(\tilde{x}\cosh(\theta) - x\sinh(\theta) - \sqrt{\frac{2\hbar}{m\omega}}\alpha_1\right)\right].
\end{aligned} \tag{36}$$

The expression (36) is just the wavefunction of the thermalized squeezed number state in the coordinate representation. When $\beta \to \infty$, the x-part of equation (36) is consistent with equation (20) in [1].

In this section, we have constructed the thermalized displaced number and squeezed number states with the thermal creation and annihilation operators, and given their wavefunctions. These states are physically meaningful. For a displaced thermalized squeezed state, Fearn and Collett gave a physical interpretation, namely that it corresponds to the output from a linear photon amplifier whose input is a squeezed state if the amplifier's added noise is regarded as thermal photons [18]. Thus, according to this interpretation, it is not difficult to give physical interpretations for the states in this paper: thermalized displaced number and squeezed number states. For instance, the thermalized displaced number state should correspond, at least theoretically, to the output from a thermal source who excites the previous output from a linear photon amplifier with a number state being the input, if no thermal noises accompany the amplifier and the thermal source can excite an electromagnetic field from its ground state to a thermal chaotic state. Of course, strictly speaking, it is impossible to have no thermal noises, and so one should consider a type of completely thermalized state in which both the displacing and the squeezing are companied by thermal noises. We shall discuss the more practical situations in a separate paper. Next, we shall discuss the time evolution of the two thermalized non-classical states.

4. Time evolution of the thermalized non-classical states

In this section, we first consider the time evolution of the thermalized displaced number and squeezed number states and then calculate their position probability densities.

In thermofield dynamics, Hamiltonian $\hat{H}$ of the combined system of the physical and tilde oscillators is [16]

$$\hat{H} = H - \tilde{H} = (a^\dagger a - \tilde{a}^\dagger \tilde{a})\hbar\omega = (a_\beta^\dagger a_\beta - \tilde{a}_\beta^\dagger \tilde{a}_\beta)\hbar\omega. \tag{37}$$

Hence the time-evolution operator of the combined system is [16]

$$U(t) = \exp\left\{-\frac{\mathrm{i}}{\hbar}\hat{H}t\right\} = \exp\{-\mathrm{i}\omega t a_\beta^\dagger a_\beta\}\exp\{\mathrm{i}\omega t\tilde{a}_\beta^\dagger \tilde{a}_\beta)\} \tag{38}$$

with time t. From equations (3), (16) and (17), we have

$$a_\beta = \frac{1}{\sqrt{2m\hbar\omega}}(\mathrm{i}p_\beta + m\omega x_\beta) \qquad a_\beta^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}(-\mathrm{i}p_\beta + m\omega x_\beta) \tag{39}$$

and

$$\tilde{a}_\beta = \frac{1}{\sqrt{2m\hbar\omega}}(-\mathrm{i}\tilde{p}_\beta + m\omega\tilde{x}_\beta) \qquad \tilde{a}_\beta^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}(\mathrm{i}\tilde{p}_\beta + m\omega\tilde{x}_\beta). \tag{40}$$

Thus, in the thermal coordinate representation, the time-evolution operator can be untangled as

$$\begin{aligned} U(t) = \frac{1}{\cos(\omega t)} &\exp\left\{-\mathrm{i}\frac{m\omega}{2\hbar}\tan(\omega t)x_\beta^2\right\}\exp\left\{-\log(\cos(\omega t))x_\beta\frac{\partial}{\partial x_\beta}\right\} \\ &\times\exp\left\{\mathrm{i}\frac{\hbar}{2m\omega}\tan(\omega t)\frac{\partial^2}{\partial x_\beta^2}\right\}\exp\left\{\mathrm{i}\frac{m\omega}{2\hbar}\tan(\omega t)\tilde{x}_\beta^2\right\} \\ &\times\exp\left\{-\log(\cos(\omega t))\tilde{x}_\beta\frac{\partial}{\partial\tilde{x}_\beta}\right\}\exp\left\{-\mathrm{i}\frac{\hbar}{2m\omega}\tan(\omega t)\frac{\partial^2}{\partial\tilde{x}_\beta^2}\right\}. \end{aligned} \tag{41}$$

The operator $U(t)$ acting on a wavefunction will yield the time evolution of the wavefunction. With the help of equations (9) and (10) and the following operator property [27]:

$$\exp\{C\partial_y^2\}f(y) = \frac{1}{\sqrt{4\pi C}}\int_{-\infty}^{\infty}\exp\left\{-\frac{(w-y)^2}{4C}\right\}f(w)\,\mathrm{d}w \tag{42}$$

which can be shown by using the identity

$$\mathrm{e}^{CQ^2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-\sigma^2/2}\mathrm{e}^{\sigma\sqrt{2C}Q}\,\mathrm{d}\sigma$$

we can perform the action of $U(t)$ on various thermalized non-classical states here.

Letting $U(t)$ equation (41) act on the wavefunction equation (14), one can find that the wavefunction equation (14) is invariant and hence the wavefunction of the thermal vacuum is independent of time, i.e., $\langle\tilde{x}, x|0, \beta, t\rangle \equiv U(t)\langle\tilde{x}, x|0, \beta\rangle = \langle\tilde{x}, x|0, \beta\rangle$. This is understandable because the average value of any physical observable on the thermal vacuum is equal to its ensemble average value, which does not vary with time.

$U(t)$ equation (41) acting on equation (26) gives the time-dependent wavefunction of the thermalized displaced number state $\langle\tilde{x}, x|\alpha, n, \beta, t\rangle \equiv U(t)\langle\tilde{x}, x|\alpha, n, \beta\rangle$ as

$$\langle\tilde{x}, x|\alpha, n, \beta, t\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}}\frac{1}{2^n n!}\exp\left\{\left(\frac{\alpha^2}{A} + \frac{\alpha^{*2}}{A^*}\right)\cos(\omega t) - 2\alpha_1^2\right\}$$

$$
\times \exp\left\{-\frac{m\omega}{2\hbar}\left[\left(x_\beta-\sqrt{\frac{2\hbar}{m\omega}}\frac{\alpha}{A}\right)^2+\left(\tilde{x}_\beta-\sqrt{\frac{2\hbar}{m\omega}}\frac{\alpha^*}{A^*}\right)^2\right]\right\}
$$
$$
\times H_n\left[\sqrt{\frac{m\omega}{\hbar}}x_\beta-\sqrt{2}(\alpha_1\cos(\omega t)+\alpha_2\sin(\omega t))\right]
$$
$$
\times H_n\left[\sqrt{\frac{m\omega}{\hbar}}\tilde{x}_\beta-\sqrt{2}(\alpha_1\cos(\omega t)+\alpha_2\sin(\omega t))\right] \tag{43}
$$

with $A=\cos(\omega t)+\mathrm{i}\sin(\omega t)$. In finishing the relevant integral with a Hermite polynomial, we used formula 7.374 (8) in [29]. Equation (43) indicates that the wavefunction of the thermalized displaced number state is dependent upon time.

Similarly, $U(t)$ equation (41) acting on equation (35) yields the time-dependent wavefunction of the thermalized squeezed number state $\langle\tilde{x}, x|\alpha, z, n, \beta, t\rangle \equiv U(t)\langle\tilde{x}, x|\alpha, z, n, \beta\rangle$ as

$$
\langle\tilde{x}, x|\alpha, z, n, \beta, t\rangle=\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}}\frac{1}{|\mathcal{F}_1|2^n n!|B|}
$$
$$
\times \exp\left\{-\frac{\mathcal{F}_2\cos(\omega t)\alpha_1^2+2\mathcal{F}_2\sin(\omega t)\alpha_1\alpha_2+\mathrm{i}\sin(\omega t)\alpha_2^2}{B}\right\}
$$
$$
\times \exp\left\{-\frac{\mathcal{F}_2^*\cos(\omega t)\alpha_1^2+2\mathcal{F}_2^*\sin(\omega t)\alpha_1\alpha_2-\mathrm{i}\sin(\omega t)\alpha_2^2}{B^*}\right\}
$$
$$
\times \exp\left\{-\frac{m\omega}{2\hbar}\frac{\mathcal{F}_2\cos(\omega t)+\mathrm{i}\sin(\omega t)}{B}x_\beta^2+2\sqrt{\frac{m\omega}{2\hbar}}\frac{\mathcal{F}_2\alpha_1+\mathrm{i}\alpha_2}{B}x_\beta\right\}
$$
$$
\times \exp\left\{-\frac{m\omega}{2\hbar}\frac{\mathcal{F}_2^*\cos(\omega t)-\mathrm{i}\sin(\omega t)}{B^*}\tilde{x}_\beta^2+2\sqrt{\frac{m\omega}{2\hbar}}\frac{\mathcal{F}_2^*\alpha_1-\mathrm{i}\alpha_2}{B^*}\tilde{x}_\beta\right\}
$$
$$
\times H_n\left[\sqrt{\frac{m\omega}{\hbar}}(\mathcal{F}_4|B|)^{-1}\left(x_\beta-\sqrt{\frac{2\hbar}{m\omega}}(\cos(\omega t)\alpha_1+\sin(\omega t)\alpha_2)\right)\right]
$$
$$
\times H_n\left[\sqrt{\frac{m\omega}{\hbar}}(\mathcal{F}_4|B|)^{-1}\left(\tilde{x}_\beta-\sqrt{\frac{2\hbar}{m\omega}}(\cos(\omega t)\alpha_1+\sin(\omega t)\alpha_2)\right)\right] \tag{44}
$$

with $B=\cos(\omega t)+\mathrm{i}\mathcal{F}_2\sin(\omega t)$. Like the thermalized displaced number state, the wavefunction of the thermalized squeezed number state is also time dependent. Substituting equations (12) and (13) into (43) and (44), one can obtain the time-dependent wavefunctions of the thermalized displaced number and squeezed number states in the coordinate representation. When $\beta\to\infty$, the x-part of equation (44) is consistent with equation (45) in [1], except for the lack of an imaginary exponent, which is cancelled by the relevant exponent from the tilde system (in equation (46) of [1] there should be the symbol '=' between the parentheses and the fraction). Additionally, for the case of $\beta\to\infty$, the x-part of equation (43) is consistent with the zero-z resultant of equation (45) in [1] (except for the lack of an imaginary exponent).

Equations (43) and (44) indicate that the wavefunctions of both the thermalized displaced number state and the thermalized squeezed number state are dependent upon time, which is different from the thermal vacuum wavefunction. This point is because both the thermalized displaced number state and the thermalized squeezed number state are not eigenstates of the Hamiltonian $\hat{H}$, while the thermal vacuum is an eigenstate of the Hamiltonian $\hat{H}$ with zero eigenvalue.

Now we can calculate the position probability densities. The probability density is the modulus square of the position-coordinate wavefunction with the tilde coordinate integrated.

First, we consider the thermal vacuum. The density is easy to calcuate and the result is

$$\rho_v(x,t) \equiv \int_{-\infty}^{\infty} \langle t,\beta,0|x,\tilde{x}\rangle\langle\tilde{x},x|0,\beta,t\rangle\,\mathrm{d}\tilde{x} = \int_{-\infty}^{\infty}\langle\beta,0|x,\tilde{x}\rangle\langle\tilde{x},x|0,\beta\rangle\,\mathrm{d}\tilde{x}$$

$$= \sqrt{\frac{m\omega}{\pi\hbar}}\tanh^{\frac{1}{2}}\left(\frac{\beta\hbar\omega}{2}\right)\exp\left\{-\frac{m\omega}{\hbar}\tanh\left(\frac{\beta\hbar\omega}{2}\right)x^2\right\}. \tag{45}$$

This is just the familiar result in statistical mechanics.

Secondly, we calculate the position probability density of the thermalized displaced number state

$$\rho_c(x,t) \equiv \int_{-\infty}^{\infty}\langle t,\beta,n,\alpha|x,\tilde{x}\rangle\langle\tilde{x},x|\alpha,n,\beta,t\rangle\,\mathrm{d}\tilde{x}. \tag{46}$$

Substituting equations (12), (13) and (43) into equation (46) and reducing it, we have

$$\rho_c(x,t) = \frac{m\omega}{\pi\hbar}\left(\frac{1}{2^n n!}\right)^2\int_{-\infty}^{\infty}\exp\{-(a_1\tilde{x}+a_2)^2-(b_1\tilde{x}+b_2)^2\}(H_n[a_1\tilde{x}+a_2])^2 \times (H_n[-b_1\tilde{x}-b_2])^2\,\mathrm{d}\tilde{x} \tag{47}$$

where,

$$a_1 = \sqrt{\frac{m\omega}{\hbar}}\cosh(\theta) \qquad a_2 = -\sqrt{\frac{m\omega}{\hbar}}x\sinh(\theta) - \sqrt{2}(\alpha_1\cos(\omega t)+\alpha_2\sin(\omega t))$$

$$b_1 = \sqrt{\frac{m\omega}{\hbar}}\sinh(\theta) \qquad b_2 = -\sqrt{\frac{m\omega}{\hbar}}x\cosh(\theta) + \sqrt{2}(\alpha_1\cos(\omega t)+\alpha_2\sin(\omega t)).$$

With the help of the formula on p 225 of [30]

$$H_m[x]H_n[x] = \sum_{r=0}^{\min\{m,n\}} 2^r r! C_m^r C_n^r H_{m+n-2r}[x] \tag{48}$$

with C_n^r and C_m^r being combinations, equation (47) can be written as

$$\rho_c(x,t) = \frac{m\omega}{\pi\hbar a_1}\left(\frac{1}{2^n n!}\right)^2\sum_{j,k=0}^{n}2^{j+k}j!k!(C_n^jC_n^k)^2\int_{-\infty}^{\infty} \times \exp\{-y^2-(ay+b)^2\}H_{2n-2j}[y]H_{2n-2k}[ay+b]\,\mathrm{d}y \tag{49}$$

with $a = \frac{b_1}{a_1}$ and $b = -aa_2 + b_2$. Using repeatedly the formula

$$\mathrm{e}^{-x^2}H_n[x] = -\frac{\mathrm{d}}{\mathrm{d}x}\{\mathrm{e}^{-x^2}H_{n-1}[x]\}$$

and then formula 7.374 (8) in [29], one can obtain

$$\begin{aligned}\rho_c(x,t) &= \frac{m\omega}{\pi\hbar a_1}\left(\frac{1}{2^n n!}\right)^2\sqrt{\frac{\pi}{a^2+1}}\exp\left\{-\frac{b^2}{a^2+1}\right\} \\ &\quad\times\sum_{j,k=0}^{n}2^{j+k}j!k!(C_n^jC_n^k)^2a^{2(n-j)}(a^2+1)^{j+k-2n}H_{2(2n-j-k)}\left[\frac{b}{\sqrt{a^2+1}}\right] \\ &= \sqrt{\frac{m\omega}{\pi\hbar}}\left(\frac{1}{2^n n!}\right)^2\tanh^{\frac{1}{2}}\left(\frac{\beta\hbar\omega}{2}\right) \\ &\quad\times\exp\left\{-\left[\sqrt{\frac{m\omega}{\hbar}\tanh\left(\frac{\beta\hbar\omega}{2}\right)}x-\sqrt{2}(\alpha_1\cos(\omega t)+\alpha_2\sin(\omega t))\right.\right.\end{aligned} \tag{50}$$

$$\times\sqrt{1+\operatorname{sech}\left(\frac{\beta\hbar\omega}{2}\right)}\Bigg]^2\Bigg\}$$

$$\times\sum_{j,k=0}^{n}2^{2(j+k)-2n}j!k!(C_n^jC_n^k)^2\exp\left\{\frac{1}{2}(j-k)\hbar\omega\beta\right\}\left(\cosh\left(\frac{\beta\hbar\omega}{2}\right)\right)^{j+k-2n}$$

$$\times H_{2(2n-j-k)}\Bigg[\sqrt{\frac{m\omega}{\hbar}\tanh\left(\frac{\beta\hbar\omega}{2}\right)}x-\sqrt{2}(\alpha_1\cos(\omega t)+\alpha_2\sin(\omega t))$$

$$\times\sqrt{1+\operatorname{sech}\left(\frac{\beta\hbar\omega}{2}\right)}\Bigg]. \tag{51}$$

As for the thermalized squeezed number state, the calculation of the position probability density is completely similar to that of the thermalized displaced number state. Finishing calculations similar to the above, one can find the position probability density of the thermalized squeezed number state equation (44)

$$\rho_s(x,t)\equiv\int_{-\infty}^{\infty}\langle\tilde{x},x|\alpha,z,n,\beta,t\rangle\langle t,\beta,n,z,\alpha|x,\tilde{x}\rangle\,\mathrm{d}\tilde{x} \tag{52}$$

$$=\frac{m\omega\mathcal{F}_4|B|}{\pi\hbar a_1}\left(\frac{1}{2^nn!}\right)^2\sqrt{\frac{\pi}{a^2+1}}\frac{(\mathcal{F}_3^*\mathcal{F}_3)^n}{\mathcal{F}_1^*\mathcal{F}_1B^*B}\exp\left\{-\frac{b^2}{(a^2+1)\mathcal{F}_4^2B^*B}\right\}$$
$$\times\sum_{j,k=0}^{n}2^{j+k}j!k!(C_n^jC_n^k)^2a^{2(n-j)}(a^2+1)^{j+k-2n}H_{2(2n-j-k)}\left[\frac{b}{\sqrt{a^2+1}\mathcal{F}_4|B|}\right] \tag{53}$$

$$=\sqrt{\frac{m\omega}{\pi\hbar}}\left(\frac{1}{2^nn!}\right)^2\frac{1}{\mathcal{F}_4|B|}\tanh^{\frac{1}{2}}\left(\frac{\beta\hbar\omega}{2}\right)$$

$$\times\exp\Bigg\{-\frac{1}{\mathcal{F}_4^2B^*B}\Bigg[\sqrt{\frac{m\omega}{\hbar}\tanh\left(\frac{\beta\hbar\omega}{2}\right)}x-\sqrt{2}(\alpha_1\cos(\omega t)$$

$$+\alpha_2\sin(\omega t))\sqrt{1+\operatorname{sech}\left(\frac{\beta\hbar\omega}{2}\right)}\Bigg]^2\Bigg\}$$

$$\times\sum_{j,k=0}^{n}2^{2(j+k)-2n}j!k!(C_n^jC_n^k)^2\exp\left\{\frac{1}{2}(j-k)\hbar\omega\beta\right\}\left(\cosh\left(\frac{\beta\hbar\omega}{2}\right)\right)^{j+k-2n}$$

$$\times H_{2(2n-j-k)}\Bigg[\frac{1}{\mathcal{F}_4|B|}\Bigg(\sqrt{\frac{m\omega}{\hbar}\tanh\left(\frac{\beta\hbar\omega}{2}\right)}x-\sqrt{2}(\alpha_1\cos(\omega t)$$

$$+\alpha_2\sin(\omega t))\sqrt{1+\operatorname{sech}\left(\frac{\beta\hbar\omega}{2}\right)}\Bigg)\Bigg]. \tag{54}$$

When $\beta\to\infty$, the existence of the factor $a^{2(n-j)}$ enforces the summation index j have a unique value n because $a=0$. Thus, employing equation (48), one has

$$\sum_{j,k=0}^{n}2^{j+k}j!k!(C_n^jC_n^k)^2a^{2(n-j)}(a^2+1)^{j+k-2n}H_{2(2n-j-k)}\left[\frac{b}{\sqrt{a^2+1}\mathcal{F}_4|B|}\right]$$
$$=2^nn!\left(H_n\left[\frac{b}{\sqrt{a^2+1}\mathcal{F}_4|B|}\right]\right)^2$$

and hence equation (54) with $\beta \to \infty$ can give equation (50) in [1]. Meanwhile, the probability density equation (51) with $\beta \to \infty$ can also lead to equation (50) with $z = 0$ in [1].

For a given time, for example, $t = 0$, one can get the position probability densities $\rho_c(x)$ and $\rho_s(x)$ without considering the time evolution from equations (51) and (54). Contrasting $\rho_c(x)$ and $\rho_c(x, t)$, as well as $\rho_s(x)$ and $\rho_s(x, t)$, one can find that only the displacement and squeeze parameters in the expressions of the densities experience changes with the evolution of time. That is, for the thermalized displaced number state, the real part of the displacement parameter α becomes $\alpha_1 \cos(\omega t) + \mathrm{i}\alpha_2 \sin(\omega t)$, which is similar to that in [22 (1965)], and for the thermalized squeezed number state, besides the same change of α_1, the parameter $\mathcal{F}_4$ becomes $\mathcal{F}_4|B|$.

The expressions of both $\rho_c(x, t)$ and $\rho_s(x, t)$ are complicated, but one can easily prove that each of them is normalized, because only the term with $j = k = n$ is not zero when $\rho_c(x, t)$ or $\rho_s(x, t)$ is integrated with respect to x. Furthermore, one can calculate the average value of the position coordinate x on the thermalized squeezed number state as

$$\langle x \rangle \equiv \int_{-\infty}^{\infty} x\rho_s(x, t) = \sqrt{\coth\left(\frac{\beta\hbar\omega}{4}\right)}\sqrt{\frac{2\hbar}{m\omega}}(\alpha_1 \cos(\omega t) + \alpha_2 \sin(\omega t)). \quad (55)$$

Evidently, $\langle x \rangle$ is independent of n and the squeeze parameter z, which is similar to the result of the squeezed number state [1]. Equation (55) indicates that the average value of x on any thermalized squeezed number state follows the motion of a classical harmonic oscillator, and the amplitude of the oscillation increases with the increase of the temperature. When $\beta \to \infty$, equation (55) is consistent with equation (36) in [1], and in contrast to equation (36) in [1], equation (55) has just an additional temperature factor $\sqrt{\coth\left(\frac{\beta\hbar\omega}{4}\right)}$.

Besides, one also easily obtain the variance of x on a thermalized squeezed number state as

$$(\Delta_n x)^2 >\equiv \int_{-\infty}^{\infty} (x - \langle x \rangle)^2 \rho_s(x, t) = \coth\left(\frac{\beta\hbar\omega}{2}\right)(2n + 1)\frac{\hbar\mathcal{F}_4^2|B|^2}{2m\omega}. \quad (56)$$

When $\beta \to \infty$, $(\Delta_n x)^2$ is consistent with equation (38) in [1]. When $n = 0$, $(\Delta_0 x)^2$ is consistent with equation (15*a*) in Kireev *et al* [19]. A comparison of equation (56) with equation (38) in [1] tells us that $(\Delta_n x)^2$ just has an additional temperature factor $\coth\left(\frac{\beta\hbar\omega}{2}\right)$.

Finally, we give two examples of $\rho_s(x, t)$ to end this section. For $n = 0$, we have

$$\rho_s(x, t) = \sqrt{\frac{1}{2\pi(\Delta_0 x)^2}} \exp\left\{-\frac{(x - \langle x \rangle)^2}{2(\Delta_0 x)^2}\right\}. \quad (57)$$

Using the relation between the thermalized squeezed state and squeezed thermalized state in [18 (1991)], one can find that the last equation with $t = 0$ is consistent with equation (6.6*a*) in [20 (1993)]. For $n = 1$, we have

$$\rho_s(x, t) = \sqrt{\frac{2}{\pi(\Delta_0 x)^2}} \exp\left\{ -\frac{(x - \langle x \rangle)^2}{2(\Delta_0 x)^2}\left\{\frac{1}{2}\mathrm{sech}^2\left(\frac{\beta\hbar\omega}{2}\right)\left[\left(\frac{(x - \langle x \rangle)^2}{2(\Delta_0 x)^2} - \frac{3}{2}\right)^2 - \frac{3}{2}\right] + \frac{(x - \langle x \rangle)^2}{2(\Delta_0 x)^2}\right\}\right\}. \quad (58)$$

From the last equation, we see that by introducing a finite temperature effect, the x-polynomial factor in the expression of $\rho_s(x, t)$ is not just the quadratic term of $(x - \langle x \rangle)$ as equation (52) in [1], but has an additional quadruplicate term of $(x - \langle x \rangle)$. The appearance of the quadruplicate term is understandable because thermalizing a non-classical state amounts to doubling the freedom number of the systems within the framework of thermofield dynamics [16].

From equation (55) to the last equation, we give some explicit results only about the thermalized squeezed number state. As for the thermalized displaced number state, taking $z = 0$ in equations (55)—(58), one can obtain the corresponding results about them, which are also consistent with those in the literature.

5. Conclusion

In this paper, we have given the wavefunctions of the thermalized displaced number and squeezed number states in the coordinate representation. Furthermore, with the help of the thermal coordinate representation, we obtain the time-dependent expressions of these wavefunctions. Although the thermal vacuum wavefunction is time independent, but either the thermalized displaced number state or the thermalized squeezed number state varies with time. We also give the probability densities, average values and variances of the position coordinate on these states. Each of the wavefunctions, the time-dependent wavefunctions, the probability densities, average values or variances here are consistent with those in the literature when the temperature tends to zero. Setting $n = 0$ in the expressions of this paper, one can obtain results of the usual thermalized coherent and squeezed states. Additionally, setting $t = 0$, one can obtain the probability densities without considering the time evolution of these states.

In the thermal coordinate representation, the forms of the wavefunctions equations (14), (26), (35), (43) and (44) resemble their own zero-temperature limits. Of course, this resemblance does not exist in the coordinate representation at all, and the probabilty densities are different from those at the zero-temperature case. From section 4, one has seen that the thermal coordinate representation greatly simplified the calculations there. Perhaps the thermal coordinate representation would simplify other calculations related to the thermal non-classical states.

Finally, although the thermalized displaced number and squeezed number states are discussed, the above results of them can easily give the corresponding ones of other similar states, such as displaced thermalized number state, squeezed thermalized number state, etc, with a simple parameter transformation [18 (1991), 20 (1993)]. Additionally, no matter how complicated expressions (51) and (54) are, it is not difficult to compute them numerically for a given number n. We believe that once the displaced number state and squeezed number state are prepared in laboratories some day, the results in this paper will be useful.

Acknowledgments

This project was supported jointly by the President Foundation of Shanghai Jiao Tong University and the National Natural Science Foundation of China with grant No 19875034.

References

[1] Nieto M M 1997 *Phys. Lett.* A **229** 135
[2] Glauber R J 1963 *Phys. Rev.* **131** 2766
[3] de Lange O L and Raab R E 1991 *Operator Methods in Quantum Mechanics* (Oxford: Clarendon)
[4] Yuen H P 1976 *Phys. Rev.* A **13** 2226
Hollenhorst J N 1979 *Phys. Rev.* D **19** 1669
[5] Caves C M, Thorne J S, Drever R W P, Sandberg V D and Zimmerman 1980 *Rev. Mod. Phys.* **52** 341
Zhang W M, Feng D H and Gilmore R 1990 *Rev. Mod. Phys.* **62** 867
[6] Klauder J R and Sleagerstam B-S (ed) 1985 *Coherent States: Applications in Physics and Mathematical Physics* (Singapore: World Scientific)
Feng D H, Klauder J R and Strayer M R (ed) 1994 *Coherent States: Past, Present and Future* (Singapore: World Scientific)

[7] de Matos Filho R L and Vogel W 1996 *Phys. Rev. Lett.* **76** 608
Monroe C, Meekhof D M, King B E and Wineland D J 1996 *Science* **272** 1131
[8] Ekert A K and Knight P L 1989 *Am. J. Phys.* **57** 692
[9] Meekhof D M, Monroe C, King B E, Itano W M and Wineland D J 1996 *Phys. Rev. Lett.* **76** 1796
James D F V 1998 *Phys. Rev. Lett.* **81** 317
[10] Ramarao I (ed) 1993 *Recent Developments in Quantum Optics* (New York: Plenum)
[11] Král P 1990 *J. Mod. Opt.* **37** 889
Král P 1990 *Phys. Rev.* **42** 4177
[12] Husimi K 1953 *Prog. Theor. Phys.* **9** 381
Senitzky I R 1954 *Phys. Rev.* **95** 1115
Plebanski J 1956 *Phys. Rev.* **101** 1825
Epstein S T 1959 *Am. J. Phys.* **27** 291
[13] Boiteux M and Levelut A 1973 *J. Physique* **6** 589
Roy S M and Singh V 1982 *Phys. Rev.* D **25** 3413
Satyanarayana M V 1985 *Phys. Rev.* D **32** 400
de Oliveira F A M, Kim M S, Knight P L and Bužek V 1990 *Phys. Rev.* A **41** 2645
de Oliveira F A M, Kim M S, Knight P L and Bužek V 1990 *J. Mod. Opt.* **37** 659
Bishop R F and Vourdas A 1994 *Phys. Rev.* A **50** 4488
Marian P 1991 *Phys. Rev.* A **44** 3325
[14] Kim M S, de Oliveira F A M and Knight P L 1989 *Opt. Commun.* **72** 99
Kim M S, de Oliveira F A M and Knight P L 1989 *Phys. Rev.* A **40** 2494
[15] Slusher R E, Hollberg L W, Yurke B, Mertz J C and Valley J F 1985 *Phys. Rev. Lett.* **55** 2409
Hong C K and Mandel L 1986 *Phys. Rev. Lett.* **56** 58
Leibfried D, Meekhof D M, King B E, Monroe C, Itano W M and Wineland D J 1996 *Phys. Rev. Lett.* **77** 4281
[16] Takahashi Y and Umezawa H 1975 *Collect. Phenom.* **2** 55
Umezawa H, Matsumoto H and Tachiki M 1982 *Thermo Field Dynamics and Condensed states* (Amsterdam: North-Holland)
Das A 1997 *Finite Temperature Field Theory* (Singapore: World Scientific)
[17] Barnett S M and Knight P L 1985 *J. Opt. Soc. Am.* B **2** 467
[18] Fearn H and Collett M J 1988 *J. Mod. Opt.* **35** 553
Oz-Vogt J, Mann A and Revzen M 1991 *J. Mod. Opt.* **38** 2339
[19] Mann A and Revzen M 1989 *Phys. Lett.* A **134** 273
Kireev A, Mann A, Revzen M and Umezawa H 1989 *Phys. Lett.* A **142** 215
Mann A, Revzen M, Umezawa H and Yamanaka Y 1991 *J. Phys. A: Math. Gen.* **24** 4527
Chaturvedi S, Sandhya R and Srinivasan V 1990 *Phys. Rev.* A **41** 3969
[20] Emch G G and Hegerfeldt G C 1986 *J. Math. Phys.* **27** 2731
Garavaglia T 1990 *Phys. Lett.* A **145** 215
Marian P and Marian T A 1993 *Phys. Rev.* A **47** 4474
Marian P and Marian T A 1996 *Ann. Phys., Lpz.* **245** 98
[21] Aliaga J and Proto A N 1989 *Phys. Lett.* A **142** 63
Aliaga J, Crespo G and Proto A N 1990 *Phys. Rev.* A **42** 618
Dodonov V V, Man'ko O V, Man'ko V I and Rosa L 1994 *Phys. Lett.* A **185** 231
[22] Lachs G 1965 *Phys. Rev.* B **138** 1012
Vourdas A 1986 *Phys. Rev.* A **34** 3466
Vourdas A and Weiner R M 1987 *Phys. Rev.* A **36** 5866
Bishop R F and Vourdas A 1987 *J. Phys. A: Math. Gen.* **20** 3743
Vourdas A and Bishop R F 1994 *Phys. Rev.* A **50** 3331
[23] Nath R and Kumar P 1996 *J. Mod. Opt.* **43** 7
[24] Marian P and Marian T A 1996 *J. Phys. A: Math. Gen.* **29** 6233
[25] Senitzky I R 1960 *Phys. Rev.* **119** 671
[26] Kirzhnits D A 1967 *Field Theoretical Methods in Many-body Systems* (New York: Pergamon)
[27] Nieto M M 1996 *Quantum Semiclass. Opt.* **8** 1061
(Nieto M M 1996 *Preprint* LA-UR-96-1596)
[28] Wen-Fa Lu 1999 *J. Phys. A: Math. Gen.* **32** 739
(Wen-Fa Lu 1998 *Preprint* hep-th/9807025)
[29] Gradshteyn I S and Ryzhik I M 1980 *Table of Integrals, Series, and Products* (New York: Academic) p 837
[30] Magnus W, Oberhettinger F and Soni R P 1966 *Formulas and Theorems for the Special Functions of Mathematical Physics* (Berlin: Springer)